# THIN ELASTIC PERIODIC PLATES WITH UNILATERAL INTERNAL CONTACTS

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A procedure is described for a formal asymptotic expansion which realizes the transition from a three-dimensional problem of the theory of elasticity in a thin layer to a problem of plate theory for bodies of periodic structure containing a system of periodically distributed contacts.

In practical applications, it is common for periodic bodies with systems of internal contacts (a body with a system of cracks occupying a fixed region was examined in [1]) to be realized in the form of thin plates and shells: reticular shells, cloths of different weaves, reinforced shells, etc. The periodicity of the structure of such bodies is the direct result of the technology used in their manufacture. The size of a cell of the structure is comparable to the thickness of the plate. The unilateral contacts – also being a result of the manufacturing technology – obviously influence the mechanical properties of the constituent materials (examples are the different stiffnesses of a woven fabric in tension and compression or the existence of a nonlinear stress–strain relation for a similarly structured mesh when made of linearly elastic materials). There is a fairly extensive literature on the question of the conversion of three-dimensional problems of the theory of elasticity to two-dimensional problems (see [2], for example). We will partially follow [3] in obtaining our expansion, while we will follow [1] (to the extent possible) in analyzing problems with unilateral constraints that are encountered in this context. In light of this, most of our attention will be focused on features of the problem that differ from the features discussed in [1, 3].

Formulation of the Problem. We will examine a linearly elastic body ($a_{ijkl}(\mathbf{x}/\varepsilon)$ is the tensor of the elastic constants) with a periodic structure. The body occupies a thin (characteristic thickness $\varepsilon \ll 1$) region $\Omega_\varepsilon$. We use $P_\varepsilon$ to denote a cell of the structure (see Fig. 1). We impose the standard conditions [4, 5] on the elastic constants: $a_{ijkl}(\mathbf{y}) \in L_\infty(R^3)$, $\|a_{ijkl}\|_{L_\infty(R^3)} < \infty$; $a_{ijkl}(\mathbf{y})e_{ij}e_{kl} \geqslant m\|\{e_{ij}\}\|^2 > 0$ for all $\{e_{ij}\} \neq 0$ (such that $e_{ij} = e_{ji}$) and for all $\mathbf{y} \in R^3$.

The formalization of the conditions of unilateral contact have the following form [1, 4]. Let the body be fastened to the surface $\Gamma_\varepsilon{}^0$ (see Fig. 1). We introduce the space of functions $V = \{\mathbf{u} \in \{H^1(\Omega_\varepsilon)\}^3\colon\ \mathbf{u}(\mathbf{x}) = 0 \text{ on } \Gamma_\varepsilon^0\}$. Then the condition of unilateral ideal contact takes the following form [1, 4] in terms of the displacements $\mathbf{u}^\varepsilon$

$$\mathbf{u}^\varepsilon \in M = \{\mathbf{u} \in V\colon\ [\mathbf{u}\cdot\mathbf{n}] \geqslant 0 \text{ on the contact surfaces}\} \tag{1}$$

($\mathbf{n}$ is a normal to the contacting surfaces). Along with (1), we require satisfaction of its analog. The latter describes the condition of unilateral contact on a cell in terms of local variables $\mathbf{y} = \mathbf{x}/\varepsilon$ [1]:

$$\widetilde{M} = \left\{\begin{matrix}\mathbf{u} \in \{H^1(P_1)\}^3\colon\ [\mathbf{u}\cdot\mathbf{n}] \geqslant 0 \text{ on the contact surfaces and } \mathbf{u}(\mathbf{y}) \\ \text{is periodic with respect to } (y_1, y_2) \in S_1\end{matrix}\right\}.$$

Here, the square brackets denote a discontinuity (see [1]); $S_1$ is the projection of the cell $P_1$ on the plane $Oy_1y_2$ (see Fig. 1); $P_1 = (1/\varepsilon)P_\varepsilon = \{\mathbf{y} = \mathbf{x}/\varepsilon\colon\ \mathbf{x} \in P_\varepsilon\}$. As is known [1], $M$ and $\tilde{M}$ are closed convex sets.

The displacements $\mathbf{u}^\varepsilon$ of the body are found from the solution of the variational inequality [1, 4]

$$\int_{\Omega_\varepsilon} \sigma_{ij}(\mathbf{u}^\varepsilon - \mathbf{v})_{i,j}\,dv - \varepsilon^a \int_{\Gamma_\varepsilon} \mathbf{g}(\mathbf{u}^\varepsilon - \mathbf{v})\,ds \geqslant -\varepsilon^b \int_{\Omega_\varepsilon} \mathbf{f}(\mathbf{u}^\varepsilon - \mathbf{v})\,dv \tag{2}$$

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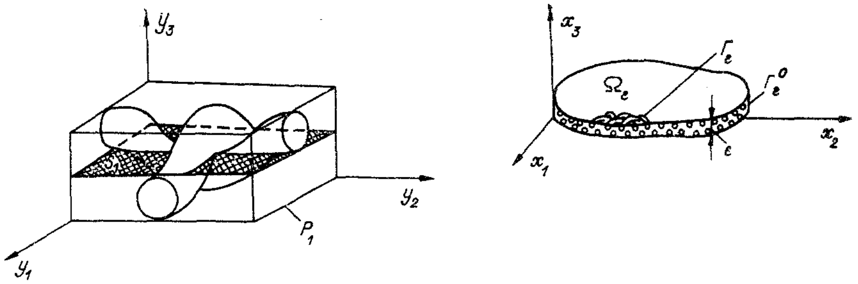


Fig. 1

for any $v \in M$, where

$$\sigma_{ij} = (1/\varepsilon^3)\, a_{ijkl}(\mathbf{x}/\varepsilon)\, u^{\varepsilon}_{k,l}. \tag{3}$$

The factors $\varepsilon^a$ and $\varepsilon^b$ in (2) determine the order of the loads on the external surfaces of the plate and the body forces. The presence of $1/\varepsilon^3$ in (3) is connected with the known estimate of the stiffness of the plate in bending [5].

Note 1. The region of contact of the elements of the given body is not known beforehand. It is determined during the solution of the problem (problem with free boundaries [1, 4]).

If the external forces are made subject to the usual conditions $\mathbf{f} \in C^1(R^3)$, $\mathbf{g} \in C^1(R^3)$, $\|\mathbf{f}\|_{C^1}, \|\mathbf{g}\|_{C^1} \leqslant m < \infty$ then problem (1)-(3) is solvable in M for any $\varepsilon > 0$ [1, 4]. We will study the problem for $\varepsilon \to 0$.

Formal Asymptotic Expansion. The formal asymptotic expansion of problem (1)-(3) will be constructed in accordance with [3] in the form

of the solution

$$\mathbf{u}^\varepsilon = \mathbf{u}^{(0)}(\tilde{\mathbf{x}}) + \varepsilon\mathbf{u}^{(1)}(\tilde{\mathbf{x}}, \mathbf{y}) + \ldots = \varepsilon^k \mathbf{u}^{(k)}; \tag{4}$$

the test function

$$\mathbf{v}^\varepsilon = \mathbf{v}^{(0)}(\tilde{\mathbf{x}}) + \varepsilon\mathbf{v}^{(1)}(\tilde{\mathbf{x}}, \mathbf{y}) + \ldots = \varepsilon^k \mathbf{v}^{(k)}; \tag{5}$$

and the stress

$$\sigma_{ij} = \frac{1}{\varepsilon^3}\sigma_{ij}^{(-3)}(\tilde{\mathbf{x}}, \mathbf{y}) + \frac{1}{\varepsilon^2}\sigma_{ij}^{(-2)}(\tilde{\mathbf{x}}, \mathbf{y}) + \ldots = \varepsilon^m \sigma_{ij}^{(m)}. \tag{6}$$

Summation is performed over repeating indices. Here, $k = 0, 1, \ldots$; $m = -3, -2, -1, 0, \ldots$; $\tilde{\mathbf{x}} = (x_1, x_2)$; $\mathbf{y} = \mathbf{x}/\varepsilon = (x_1/\varepsilon, x_2/\varepsilon, x_3/\varepsilon)$. All of the functions in the right sides (4)-(6) are assumed to be periodic with respect to $y_1$, $y_2$, with the cell $S_1$. We designate $\mathbf{w} = \mathbf{u}^\varepsilon - \mathbf{v}$. This function can be represented in the form $\mathbf{w} = \varepsilon^k \mathbf{w}^{(k)}$ $(\mathbf{w}^{(k)} = \mathbf{u}^{(k)} - \mathbf{v}^{(k)})$. Let us insert (4)-(6) into (2), (3). Changing over to the variables $\tilde{\mathbf{v}} = (x_1, x_2, x_3/\varepsilon)$, [in which the region $\Omega_\varepsilon$, with a thickness of the order of $\varepsilon$, becomes the region $\Omega_1 = \{(x_1, x_2, y_3 = x_3/\varepsilon): \mathbf{x} \in \Omega_\varepsilon\}$, with a thickness of the order of unity] and considering that for functions of the variables $\tilde{\mathbf{x}}$, $\mathbf{y}$ the differentiation operators $\partial/\partial x_i$ become $\partial/\partial x_\alpha + (1/\varepsilon)\partial/\partial y_\alpha$ when $\alpha = 1, 2$ and become $(1/\varepsilon)\partial/\partial y_3$ when $i = 3$, we obtain the following

$$\varepsilon\int_{\Omega_1} \{\varepsilon^m \sigma_{i\alpha}^{(m)} \varepsilon^k w_{i,\alpha x}^{(k)} + \varepsilon^m \sigma_{ij}^{(m)} \varepsilon^{k-1} w_{i,jy}^{(k)}\}\, d\tilde{\mathbf{v}} - \\ - \varepsilon^a \int_\Gamma \langle \mathbf{g}\varepsilon^k \mathbf{w}^{(k)}\rangle_\gamma d\tilde{\mathbf{x}} \geqslant -\varepsilon\int_{\Omega_1} \varepsilon^b \mathbf{f}\varepsilon^k \mathbf{w}^{(k)}\, d\tilde{\mathbf{v}} \tag{7}$$

($m = -3, -2, \ldots$, $k = 0, 1, \ldots$). The symbols $,\alpha x$ and $,jy$ denote differentiation with respect to $x_\alpha$ and $y_j$, respectively. Here and below, the Greek-letter subscripts take values of 1 and 2, while the Roman-letter subscripts take values of 1, 2, 3.

Note 2. In the variables $\mathbf{y} = \mathbf{x}/\varepsilon$ the cell $P_\varepsilon$ becomes a cell $P_1 = (1/\varepsilon)P_\varepsilon$ of fixed size.

Insertion of (4), (6) into governing relations (3), with allowance for the above differentiation rule, yields the equality [3]

$$\sigma_{ij}^{(m)} = a_{ijk\alpha}(\mathbf{y})\, u_{k,\alpha x}^{(m+3)} + a_{ijkl}(\mathbf{y})\, u_{k,ly}^{(m+4)} \quad (m = -3, -2, \ldots). \tag{8}$$

Let us proceed to the analysis of (7) and (8). We will do this by studying the problem with different k and m and a suitably chosen test function v in (7).

A. We take k = 0, i.e., $\mathbf{w}$ has the form $\mathbf{w} = \mathbf{w}^{(0)}(\tilde{\mathbf{x}}) \in V$ [1] (in (4), (5), $\mathbf{u}^{(1)} = \mathbf{v}^{(1)}$, $\mathbf{u}^{(2)} = \mathbf{v}^{(1)}$, etc.). By virtue of the fact that $w_{i,jy} = 0$ for the given function, inequality (7) is written in the form

$$\int\limits_{\Omega_1} \varepsilon^{m+1}\sigma_{i\alpha}^{(m)} w_{i,\alpha x}^{(0)}\, d\tilde{\mathbf{v}} - \varepsilon^{a} \int\limits_{\Gamma} \langle \mathbf{g}\mathbf{w}^{(0)}\rangle_{\gamma}\, d\tilde{\mathbf{x}} \geqslant - \int\limits_{\Omega_1} \varepsilon^{b+1}\mathbf{f}\mathbf{w}^{(0)}\, d\tilde{\mathbf{v}} \tag{9}$$

(m = −3, −2, ...) for all $\mathbf{w}^{(0)} \in V$.

Note 3. The functions $\sigma_{i\alpha}^{(m)}(\tilde{\mathbf{x}}, \mathbf{x}/\varepsilon)$ oscillate rapidly with regard to $\mathbf{x}$ in the second position. In connection with this [3],

$$\lim_{\varepsilon\to 0} \int\limits_{\Omega_1} \sigma_{i\alpha}^{(m)}(\tilde{\mathbf{x}}, \mathbf{x}/\varepsilon)\, d\tilde{\mathbf{v}} = \int\limits_{\Gamma} \langle \sigma_{i\alpha}^{(m)}\rangle (\tilde{\mathbf{x}})\, d\tilde{\mathbf{x}},$$

where $\langle\cdot\rangle = \frac{1}{\operatorname{mes} S_1}\int\limits_{P_1} d\mathbf{y}\,\cdot$ is the mean over the cell $P_1$ in the variables $\mathbf{y}$.

Then with allowance for the fact that $\mathbf{w}^{(0)}(\tilde{\mathbf{x}}) \in V$ and that V is a space, expression (9) gives the following (this case was examined in greater detail in [1])

$$\begin{aligned} &\text{at } m = -3 \ \langle \sigma_{i\alpha}^{(-3)}\rangle_{,\alpha x} = 0,\\ &\text{at } m = -2 \ \langle \sigma_{i\alpha}^{(-2)}\rangle_{,\alpha x} = 0,\\ &\text{at } m = -1 \ \langle \sigma_{i\alpha}^{(-1)}\rangle_{,\alpha x} + \langle g_i\rangle_{\gamma} = \langle f_i\rangle. \end{aligned} \tag{10}$$

In obtaining (10), we assumed that b = − 1 and $a$ = 0. Here, $\langle\cdot\rangle_{\gamma} = \frac{1}{\operatorname{mes} S_1}\int\limits_{\gamma} d\mathbf{y}$ is the mean over the free (lateral) surfaces $\gamma$ of the cell $P_1$. In deriving (10), we equated only those terms with nonpositive powers of $\varepsilon$.

B. Now we take k = 1 in (7) and we write the test function in the form

$$\mathbf{w} = \varepsilon\mathbf{w}^{(1)}(\tilde{\mathbf{x}}, \mathbf{y}) = \varepsilon y_3 \mathbf{v}_0(\tilde{\mathbf{x}}),\ \mathbf{v}_0 \in V \tag{11}$$

i.e., $\mathbf{u}^{(0)} = \mathbf{v}^{(0)}$, $\mathbf{v}^{(1)} = \mathbf{u}^{(1)} + y_3\mathbf{v}_0(\tilde{\mathbf{x}})$, $\mathbf{u}^{(2)} = \mathbf{v}^{(2)}$, etc. We obtain

$$\int\limits_{\Omega_1} \{\varepsilon^{m+2}\sigma_{i\alpha}^{(m)} y_3 v_{0i,\alpha x} + \varepsilon^{m+1}\sigma_{ij}^{(m)}\delta_{3j}v_{0i}\}\, d\tilde{\mathbf{v}} - \int\limits_{\Gamma} \varepsilon^{a+1}\langle \mathbf{g} y_3 \mathbf{v}_0\rangle_{\gamma}\, d\tilde{\mathbf{x}} \geqslant - \int\limits_{\Omega_1} \varepsilon^{b+2}\mathbf{f} y_3 \mathbf{v}_0\, d\tilde{\mathbf{v}}. \tag{12}$$

We represent the moments as $M_{ij}^{(m)} = \frac{1}{\operatorname{mes} S_1}\int\limits_{P_1} \sigma_{ij}^{(m)} y_3\, d\mathbf{y}$ [3]. By virtue of the fact that $\mathbf{v}_0(\tilde{\mathbf{x}}) \in$ V, with V being a space, we obtain the following from (12) for nonpositive powers of $\varepsilon$

$$\text{at } m = -3 \quad -M_{i\alpha,\alpha x}^{(-3)} + \langle \sigma_{i3}^{(-2)}\rangle = 0,\ \langle \sigma_{i3}^{(-3)}\rangle = 0; \tag{13}$$

$$\text{at } m = -2 \quad -M_{i\alpha,\alpha x}^{(-2)} + \langle \sigma_{i3}^{(-1)}\rangle = 0. \tag{14}$$

Note 4. At b = −1, $a$ = 0, the surface $\varepsilon^{a}\mathbf{g}$ and body $\varepsilon^{b}\mathbf{f}$ forces make no contribution in (14). Terms corresponding to them may appear for other values of $a$, b.

C. Now let us examine a local problem which arises when we put m = −3 and k = 1 in (7) and choose the test function in the form $\mathbf{w} = \varepsilon\mathbf{w}^{(1)}(\mathbf{y})$ (w is a function on $S_1$ which is periodic with respect to $y_1$ and $y_2$ and for which $w_{i,\alpha x}^{(1)} = 0$):

$$\int_{\Omega_1} \varepsilon^{-1}\sigma_{ij}^{(-3)} w_{i,jy}^{(1)}\, d\tilde{v} \geqslant 0 \quad \text{for any} \quad \mathbf{v}^{(1)} \in M. \tag{15}$$

We recall that $\mathbf{w}^{(1)} = \mathbf{u}^{(1)} - \mathbf{v}^{(1)}$. We take Eq. (8), which at m = −3 takes the form $\sigma_{ij}^{(-3)} = a_{ijk\alpha}(\mathbf{y})\, u_{k,\alpha x}^{(0)} + a_{ijkl}(\mathbf{y})\, u_{k,ly}^{(1)}$, and insert it into (15). Since the periodicity of the functions causes integration over $\Omega_1$ to reduce to integration over $P_1$ (see Note 3), we have

$$\int_{P_1} \sigma_{ij}^{(-3)} (\mathbf{u}^{(1)} - \mathbf{v}^{(1)})_{i,jy}\, d\mathbf{y} \geqslant 0 \quad \text{for any} \quad \mathbf{v}^{(1)} \in \widetilde{M}; \tag{16}$$

$$\sigma_{ij}^{(-3)} = a_{ij\beta\alpha}\gamma_{\beta\alpha}(\mathbf{u}^{(0)}) + a_{ij3\alpha} u_{3,\alpha x}^{(0)} + a_{ijkl} u_{k,ly}^{(1)}, \tag{17}$$

where $\gamma_{\beta\alpha}(\mathbf{u}^{(0)}) = (1/2)(u_{\beta,\alpha x}{}^{(0)} + u_{\alpha,\beta x}{}^{(0)})$ are the mean strains (i.e., the corresponding averaged displacements $\mathbf{u}^{(0)}$) in the plane of the plate. Problem (16) is a cellular problem (on the cell $P_1$) of "zeroth order" in the sense of [3].

In the study of the linear problem in [3], an important role was played by the fact that the solution could have been obtained in the explicit form of a cellular problem with the absolute term $a_{ij3\alpha} u_{3,\alpha x}^{(0)}$ (17). The problem is nonlinear in the given case and no solution is found in explicit form. Thus, the problem must be solved by a method different than that used in [3]. We represent the solution of problem (16) as

$$u_k^{(1)} = \tilde{u}_k^{(1)} - \delta_{k\alpha} y_3 u_{3,\alpha x}^{(0)}(\tilde{\mathbf{x}}). \tag{18}$$

Having inserted (18) into (16), we obtain

$$\int_{P_1} \sigma_{ij}^{(-3)} (\tilde{\mathbf{u}} - \mathbf{v}^{(1)})_{i,jy}\, d\mathbf{y} - \int_{P_1} \sigma_{ij}^{(-3)} u_{3,\alpha x}^{(0)} \delta_{i\alpha}\delta_{j3}\, d\mathbf{y} \geqslant 0$$

for any $\mathbf{v}^{(1)} \in \tilde{M}$. This can be rewritten in the form

$$\int_{P_1} \sigma_{ij}^{(-3)} (\tilde{\mathbf{u}} - \mathbf{v}^{(1)})_{i,jy}\, d\mathbf{y} - \int_{P_1} \sigma_{\alpha 3}^{(-3)} u_{3,\alpha x}^{(0)}\, d\mathbf{y} \geqslant 0 \tag{19}$$

for any $\mathbf{v}^{(1)} \in \tilde{M}$. By virtue of the second equality in (10), it follows from (19) that

$$\int_{P_1} \sigma_{\alpha 3}^{(-3)} u_{3,\alpha x}^{(0)}(\tilde{\mathbf{x}})\, d\mathbf{y} = \operatorname{mes} S_1 \langle \sigma_{\alpha 3}^{(-3)} \rangle u_{3,\alpha x}^{(0)} = 0.$$

After this, (19) takes the form

$$\int_{P_1} \sigma_{ij}^{(-3)} (\tilde{\mathbf{u}} - \mathbf{v}^{(1)})_{i,jy}\, d\mathbf{y} \geqslant 0 \quad \text{for any} \quad \mathbf{v}^{(1)} \in \widetilde{M}. \tag{20}$$

Substitution of $\mathbf{u}^{(1)}$, into (17) in accordance with (18) yields

$$\sigma_{ij}^{(-3)} = a_{ij\beta\alpha}\gamma_{\beta\alpha}(\mathbf{u}^{(0)}) + a_{ij3\alpha} u_{3,\alpha x}^{(0)} + a_{ijkl} u_{k,ly}^{(1)} = a_{ij\beta\alpha}\gamma_{\beta\alpha}(\mathbf{u}^{(0)}) + \\ + a_{ij3\alpha}\tilde{u}_{3,\alpha x}^{(0)} + a_{ijkl}\tilde{u}_{k,ly}^{(1)} - a_{ij3\alpha} u_{3,\alpha x}^{(0)} + 0 = a_{ij\beta\alpha}\gamma_{\beta\alpha}(\mathbf{u}^{(0)}) + a_{ijkl}\tilde{u}_{k,ly}^{(1)}. \tag{21}$$

As a result, having inserted (21) into (20), we have the problem of determining

$$\int_{P_1} \{a_{ij\beta\alpha}\gamma_{\beta\alpha}(\mathbf{u}^{(0)}) + a_{ijkl}\tilde{u}_{k,ly}^{(1)}\} (\tilde{\mathbf{u}}^{(1)} - \mathbf{v}^{(1)})_{i,jy}\, d\mathbf{y} \geqslant 0 \tag{22}$$

for any $\mathbf{v}^{(1)} \in \tilde{M}$. Variational inequality (22) is similar to that studied in [1] in an examination of a body with a system of cracks. The difference is that free surfaces $\gamma$ of the cell $P_1$ are made subject to the condition $(a_{ij\beta\alpha}\gamma_{\beta\alpha}(\mathbf{u}^0) + a_{ijkl}\tilde{u}_{k,lu}^{(1)}) n_j = 0$, (where $\mathbf{n}$ is a normal to $\gamma$), rather than to the condition of periodicity with respect to $y_3$ (the conditions of periodicity with respect to $y_1$ and $y_2$ remain in force). We will study the function obtained by averaging (21) over $P_1$:

$$\gamma_{\beta\alpha}(\mathbf{u}^{(0)}) \rightarrow \langle a_{ij\beta\alpha}\gamma_{\beta\alpha}(\mathbf{u}^{(0)}) + a_{ijkl}\tilde{u}_{k,ly}^{(1)} \rangle \equiv \Phi_{\beta\alpha}(\mathbf{u}^{(0)}) \tag{23}$$

[ $\tilde{u}^{(1)}$ is solution (22)]. The function (23) gives the governing relations in the plane of the plate. Similarly to [1] (the above-noted difference from [1] is not important in the present case), we find that: a) (23) is a hyperelastic law; b) the problem $\langle\sigma_{i\alpha}^{(-3)}\rangle_{,\alpha x} = \Phi_{\beta\alpha}(\mathbf{u}^{(0)})_{,\alpha x} = 0$ [see (10)] with the boundary condition $u_\alpha^{(0)}(\tilde{\mathbf{x}}) = 0$ on $\partial\Gamma$ has a unique (zero) solution. As a result, $u_\alpha^{(0)}(\tilde{\mathbf{x}}) = 0$ ($\alpha = 1, 2$). From this [see (18)]

$$u_\beta^{(1)} = -y_3\delta_{\beta\alpha}u_{3,\alpha x}^{(0)} + \breve{u}_\beta(\tilde{\mathbf{x}}) \quad (\alpha, \beta = 1, 2); \tag{24}$$

$$u_3^{(1)} = \breve{u}_3(\tilde{\mathbf{x}}). \tag{25}$$

D. Now we will examine the case m = −2 in (8). Here,

$$\sigma_{ij}^{(-2)} = a_{ijkl}(\mathbf{y})\,u_{k,ly}^{(2)} + a_{ijk\alpha}(\mathbf{y})\,u_{k,\alpha x}^{(1)}.$$

Having inserted the expression for $\mathbf{u}^{(1)}$, in accordance with (24)-(25) we obtain

$$\sigma_{ij}^{(-2)} = a_{ijkl}u_{k,ly}^{(2)} + a_{ij\beta\alpha}\{-y_3u_{3,\beta x}^{(0)} + \breve{u}_\beta\}_{,\alpha x} + a_{ij3\alpha}\breve{u}_{3,\alpha x} = a_{ijkl}u_{k,ly}^{(2)} - a_{ij\beta\alpha}y_3u_{3,\beta x\alpha x}^{(0)} + a_{ijk\alpha}\breve{u}_{k,\alpha x}. \tag{26}$$

Now we set m = −2, k = 2 in (7) and we take $\mathbf{w} = \varepsilon^2\mathbf{w}^{(2)}(\mathbf{y})$ ($\mathbf{w}$ is a function which is periodic with respect to $y_1$, $y_2$). Changing from integration over $\Omega_1$ to integration over $P_1$, by virtue of the periodicity of the functions in (7) [1] we have

$$\varepsilon^{-1}\int_{P_1}\sigma_{ij}^{(-2)}w_{i,jy}^{(2)}\,d\mathbf{y} \geqslant 0 \quad \text{for any} \quad \mathbf{v}^{(2)} \in \widetilde{M}. \tag{27}$$

Insertion of (26) into (27) leads to the inequality

$$\int_{P_1}\{a_{ijkl}u_{k,ly}^{(2)} - a_{ij\beta\alpha}y_3u_{3,\alpha x\beta x}^{(0)} + a_{ijk\alpha}\breve{u}_{k,\alpha x}\}\times(\mathbf{u}^{(2)} - \mathbf{v}^{(2)})_{i,jy}\,d\mathbf{y} \geqslant 0 \quad \text{for any} \quad \mathbf{v}^{(2)} \in \widetilde{M}. \tag{28}$$

Here, we again encounter a situation similar to that discussed in connection with inequality (16), due to the nonlinearity of the problem. We proceed as follows. First we introduce the function

$$\tilde{u}_i^{(2)} = u_i^{(2)} - y_3\delta_{i\alpha}\breve{u}_{3,\alpha x}. \tag{29}$$

Having inserted (29) into (28), we obtain

$$\int_{P_1}\{a_{ijkl}\tilde{u}_{k,ly}^{(2)} - a_{ij\beta\alpha}y_3u_{3,\alpha x\beta x}^{(0)} + a_{ij\beta\alpha}\breve{u}_{\beta,\alpha x}\}\times(\tilde{\mathbf{u}}^{(2)} - \mathbf{v}^{(2)})_{i,jy}\,d\mathbf{y} + \operatorname{mes} S_1\langle\sigma_{\alpha 3}^{(-2)}\rangle(\tilde{\mathbf{x}})\,\breve{u}_{3,\alpha x}(\tilde{\mathbf{x}}) \geqslant 0. \tag{30}$$

The last term in the left side of (30) is equal to zero, in accordance with the second equality in (10). Thus,

$$\int_{P_1}\{a_{ijkl}\tilde{u}_{k,ly}^{(2)} - a_{ij\beta\alpha}y_3u_{3,\alpha x\beta x}^{(0)} + a_{ij\beta\alpha}\gamma_{\beta\alpha}(\breve{\mathbf{u}})\}\,(\tilde{\mathbf{u}}^{(2)} - \mathbf{v}^{(2)})_{i,jy}\,d\mathbf{y} \geqslant 0 \tag{31}$$

for any $\mathbf{v}^{(2)} \in \tilde{M}$.

Inequality (31), being a cellular problem of the "first order" in the sense of [3] and having a unique solution [1], determines a function which places the solution of cellular problem (31) in correspondence with the quantities $\gamma_{\alpha\beta}$, $\rho_{\alpha\beta}$:

$$(\gamma_{\alpha\beta}, \rho_{\alpha\beta} \equiv u_{3,\alpha x\beta x}^{(0)}) \to \tilde{\mathbf{u}}^{(2)} \equiv \Psi(\mathbf{y}, \gamma_{\alpha\beta}, \rho_{\alpha\beta}). \tag{32}$$

By virtue of (29), (32)

$$u_i^{(2)} = \Psi_i(\mathbf{y}, \gamma_{\alpha\beta}, \rho_{\alpha\beta}) + y_3\delta_{i\alpha}\breve{u}_{3,\alpha x}. \tag{33}$$

If we substitute (33) into (26) — which coincides with the result of the substitution of $\tilde{\mathbf{u}}^{(2)}$ (29) into the expression in brackets in (31) — we obtain

$$\sigma_{ij}^{(-2)} = a_{ijkl}\Psi_k(\mathbf{y}, \gamma_{\alpha\beta}, \rho_{\alpha\beta})_{,ly} - a_{ij\beta\alpha}y_3\rho_{\beta\alpha} + a_{ij\beta\alpha}\gamma_{\beta\alpha}. \tag{34}$$

Averaging (34) over the cell $P_1$ leads to the function

$$(\gamma_{\alpha\beta}, \rho_{\alpha\beta}) \to \langle\sigma_{ij}^{(-2)}\rangle = \langle a_{ijkl}\Psi_k(\mathbf{y}, \gamma_{\alpha\beta}, \rho_{\alpha\beta})_{,ly} - a_{ij\alpha\beta}y_3\rho_{\alpha\beta} + a_{ij\beta\alpha}\gamma_{\beta\alpha}\rangle \equiv \Xi_{\alpha\beta}(\gamma_{\alpha\beta}, \rho_{\alpha\beta}). \tag{35}$$

Having multiplied (34) by $y_3$ and having averaged the resulting equality over $P_1$, we find that

$$M_{ij}^{(-2)} = \langle\sigma_{ij}^{(-2)}y_3\rangle = \langle y_3 a_{ijkl}\Psi_k(\mathbf{y}, \gamma_{\alpha\beta}, \rho_{\alpha\beta})_{,ly} - y_3^2 a_{ij\alpha\beta}\rho_{\alpha\beta} + y_3 a_{ij\beta\alpha}\gamma_{\beta\alpha}\rangle \equiv \Lambda_{\alpha\beta}(\gamma_{\alpha\beta}, \rho_{\alpha\beta}). \tag{36}$$

Let us write out some of the relations that are obtained. From (10) we have

$$\text{at } m = -2 \quad \langle\sigma_{i\alpha}^{(-2)}\rangle_{,\alpha x} = 0; \tag{37}$$

$$\text{at } m = -1 \quad \langle\sigma_{3\alpha}^{(-1)}\rangle_{,\alpha x} - \langle g_3\rangle_\gamma = \langle f_3\rangle; \tag{38}$$

while from (14), (35), and (36) we obtain

$$-M_{\beta\alpha,\alpha x}^{(-2)} + \langle\sigma_{\beta 3}^{(-1)}\rangle = 0; \tag{39}$$

$$\langle\sigma_{\alpha\beta}^{(-2)}\rangle = \Xi_{\alpha\beta}(\gamma_{\alpha\beta}, \rho_{\alpha\beta}); \tag{40}$$

$$M_{\alpha\beta}^{(-2)} = \Lambda_{\alpha\beta}(\gamma_{\alpha\beta}, \rho_{\alpha\beta}). \tag{41}$$

Here, $\gamma_{\alpha\beta} = (1/2)(\check{u}_{\alpha,\beta x} + \check{u}_{\beta,\alpha x})$ are the strains in the plane of the plate; $\rho_{\alpha\beta} = u_{3,\alpha x\beta x}^{(0)}$ are the curvatures of the plate (more accurately, the curvatures of its limiting surface).

<u>Boundary Conditions.</u> The boundary conditions follow from the initial boundary conditions $\mathbf{u}^\varepsilon(\mathbf{x}) = 0$ on $\Gamma_\varepsilon^0$, which assume the following form by virtue of (24)–(25):

$$u_3^{(0)}(\tilde{\mathbf{x}}) = 0, \quad \frac{\partial u_3^{(0)}}{\partial n}(\tilde{\mathbf{x}}) = 0, \quad \check{u}_1(\tilde{\mathbf{x}}) = \check{u}_2(\tilde{\mathbf{x}}) = 0 \text{ on } \partial\Gamma \tag{42}$$

($\Gamma$ is the projection of the region $\Omega_\varepsilon$ on the plane $Ox_1x_2$; it is independent of $\varepsilon$).

Equations (37)-(39) are the equilibrium equations of plate theory; (40) and (41) are the governing relations. It can be seen from the above that Eqs. (40)-(41), found on the basis of the solution of cellular problems, are nonlinear functions of the strains $\gamma_{\alpha\beta}$ and curvatures $\rho_{\alpha\beta}$. In this case, (40) describes the properties of the plate in its plane, while (41) describes the properties of the plate in bending.

<u>Note 5.</u> In the absence of unilateral contacts, Eqs. (37)-(41) become the relations found in [3] for a solid plate.

The above-described formal expansion can be substantiated by the methods proposed in [1, 2, 6].

The results obtained here make it possible to derive formulas to calculate the averaged characteristics of grids, fabrics, etc. It should be noted that the use of the approximate method of solving cellular problems described in [7-10] is effective for these types of materials.

## LITERATURE CITED


1. E. Sanchez-Palencia, Inhomogeneous Media and the Theory of Vibrations [Russian translation], Mir, Moscow (1984).
2. A. L. Kalamkarov, B. A. Kudryavtsev, and V. Z. Parton, "Asymptotic averaging method in the mechanics of composites with a regular structure," Itogi Nauki Tekh. Mekh. Deform. Tverd. Tela, 19, VINITI, Moscow (1987).
3. D. Gaillerie, "Thin elastic and periodic plates," Math. Meth. Appl. Sci., No. 6 (1984).
4. G. Duvant and J. L. Lions, Inequalities in Mechanics and Physics, Springer-Verlag, New York (1975).

5. Yu. N. Rabotnov, Mechanics of Deformable Solids [in Russian], Nauka, Moscow (1979).
6. G. P. Panasenko, "Averaged systems of equation of the theory of elasticity for a non-uniform plate," Usp. Mat. Nauk, 40, No. 5 (1985).
7. A. G. Kolpakov, "Determination of the average characteristics of elastic frameworks," Prikl. Mat. Mekh., 49, No. 6 (1985).
8. A. G. Kolpakov, "Mechanics of composite frameworks," Sixth National Congress on Theoretical and Applied Mechanics, Rezyumet, Varna (1989).
9. B. D. Annin, A. L. Kalamkarov, and A. G. Kolpakov, "Analysis of local stresses in fiber composites," in: Localized Damage Computer-Aided Assessment and Control, Vol. 2, Comput. Mech. Publ., Southampton (1990).
10. A. G. Kolpakov, "Dependence of velocity of elastic waves in composite media on initial stresses," 11th World Congr. on Computational Mechanics, Stuttgart, 1990: Extended Abstracts of Lectures.

# OPTIMIZATION OF THE STRUCTURE OF A VIBRATION SHIELD UNDER THE INFLUENCE OF A CONCENTRATED HARMONIC LOAD

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When waves strike the interface between media with different physicomechanical properties, a system of reflected and refracted waves is formed in the laminated medium. By changing the number, size, and material of the layers, it is possible to control the intensity of the spectrum of the wave process. There naturally arises the problem of optimizing the structure of the laminated medium with different optimization criteria and different constraints on the characteristics of the wave process. Several studies [1-5] have examined aspects of optimization of the structure of multilayered sound-reflecting shields when the materials of the layers are chosen from a certain group. Investigators have examined both the case of normal incidence of an acoustic plane wave and oblique incidence. If neither the number nor the arrangement of the constituent materials is specified beforehand, then the optimization problem is formulated within the framework of the theory of optimum control. Pontryagin's maximum principle and variational methods have been used to derive the necessary optimization conditions and construct algorithms for numerical calculations. The same methods, generalized in [5], have also been used to optimize the design of a freely oscillating laminated thick-walled sphere of minimum weight [6], in several problems involving the static thermoelasticity of thick-walled spherical vessels [7, 8], and in the design of laminated thermal insulation [5, 9, 10] and wave-type electromagnetic filters [2]. In each of these studies, the spectral characteristics of the wave process depended on one space variable and were described by ordinary differential equations.

In the present study, we examine the steady vibration of a plane elastic laminated shield which is rigidly connected to an elastic half-space and is subjected to a concentrated harmonic load. We need to optimize the structure of the shield so as to minimize total wave-energy flux in the half-space. The spectral characteristics of the wave process will depend on two space variables and will be described by partial differential equations. By using the Hankel transform [11] with respect to the radial coordinate, it is possible to formulate the corresponding optimization problem for transforms that can be described by a system of ordinary differential equations. We obtain the necessary optimization conditions, propose an algorithm, and present examples of numerical calculations.

1. Formulation of the Problem. We will examine the steady-state vibration of an elastic laminated shield of thickness $l > 0$. The shield is rigidly connected to an elastic half-space $z > l$, and is subjected to a concentrated harmonic force (see Fig. 1). Choosing from a finite number of elastic materials, we need to synthesize a laminated shield occupying the region $0 \le z \le l$. The shield must be designed so as to minimize the total energy

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